A quick run through of 180 years of Classical Mechanics – for better appreciation of Quantum Mechanics

Lagrangian and Hamiltonian Mechanics

For those who followed the course pattern of University Physics I and Classical Mechanics I or equivalents, this will largely be a review. For those haven't taken Lagrangian and Hamiltonian Mechanics, don't worry – just open up your mind, relax and absorb.

Second Course: Lagrangian and Hamiltonian Mechanics

Back to basics $m\dfrac{d^2x}{dt^2} = -kx$ [Equation of Motion]

$$\Rightarrow \frac{d}{dt}\underbrace{\left(m\frac{dx}{dt}\right)} = -kx \quad \text{[Done Nothing!]}$$

some of you recognize this as the momentum [but hold on]

$$\Rightarrow \frac{d}{dt}(m\dot{x}) = -kx \quad \left[\text{Note: } \dot{x} = \frac{dx}{dt} = v\right]$$

$$\Rightarrow \frac{d}{dt}\left(\frac{d}{d\dot{x}}\underbrace{\left(\tfrac{1}{2}m\dot{x}^2\right)}_{K}\right) = -\frac{d}{dx}\underbrace{\left(\tfrac{1}{2}kx^2\right)}_{V}$$

Hinted at obtaining **Equation of Motion** by ***taking derivatives of energy***

$$\frac{d}{dt}\left(\frac{d}{d\dot{x}}\left(\frac{1}{2}m\dot{x}^2\right)\right) = -\frac{d}{dx}\left(\frac{1}{2}kx^2\right)$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial}{\partial\dot{x}}\underbrace{\left[\frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2\right]}_{K-V \equiv L(x,\dot{x})}\right) = \frac{\partial}{\partial x}\underbrace{\left[\frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2\right]}_{K-V \equiv L(x,\dot{x})}$$

Q: Why partial derivatives?

$$\Rightarrow \boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{x}}\right) = \frac{\partial L}{\partial x}}$$ is the Equation of Motion

with $L(x,\dot{x}) = K - V$ (A new function of the dimension of energy)

kinetic energy

potential energy

$= \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$ (harmonic oscillator)

$\boxed{\text{Lagrangian}}$

(Hamilton called it the Lagrangian)

This is the essence of Lagrangian Mechanics

The Leading character in Lagrangian Mechanics

is the "Lagrangian"

$L(x, \dot{x})$ or $L(x_1, \dot{x}_1; x_2, \dot{x}_2; \cdots)$

could have several coordinates

or more generally

$$L(x_1, \dot{x}_1; x_2, \dot{x}_2; \cdots ; t)$$

may also depend explicitly on time

Recipe of **Lagrangian Mechanics**

Lagrange (1736 – 1813) wrote the book *Mécanique analytique* (1787) that defined the subject *Analytical Mechanics*

- Given system, identify co-ordinate(s)
- Write down $L = K - V$
- Plug $L(x, \dot{x})$ into $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$ (Euler-Lagrange Equation)

  then the equation of motion <u>emerges</u> <u>automatically</u>
- Generally, $L(x_1, \dot{x}_1; x_2, \dot{x}_2; \cdots)$ or $L(q_1, \dot{q}_1; q_2, \dot{q}_2; \cdots)$ or $L(\{q_i, \dot{q}_i\})$

  several coordinates

  Plug L into $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$ [an equation for each q_i]

  gives the equations of motion!

That was a ***defining moment in physics***! Let's see why

Q: Why ***Lagrangian formulation***?

Good Stuff #1:

- Energy (scalar) easier to handle from forces (vectors)

Good Stuff #2:

- Easy identification of ***coordinate-momentum pair***(s)
  - ***In QM, such pair becomes conjugated pair of operators***
- Systematic way of getting **equation of motion**

- Easy identification of ***coordinate-momentum pair***(s)

$$L = (x, \dot{x})$$

$$\underbrace{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)}_{\text{defines momentum}} - \frac{\partial L}{\partial x} = 0$$

$\frac{\partial L}{\partial x}$: coordinate x

defines momentum corresponding to coordinate x [called conjugate momentum] (momentum conjugated to coordinate x)

Define momentum $\boxed{p_x \equiv \frac{\partial L}{\partial \dot{x}}}$

How momentum evolves in time? $\boxed{\frac{d}{dt} p_x = \frac{\partial L}{\partial x}}$

$$L = (\{q_i, \dot{q}_i\}) \text{ in general}$$

$$\underbrace{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)}_{\text{defines momentum}} - \frac{\partial L}{\partial q_i} = 0$$

defines momentum conjugated to coordinate q_i

$\boxed{p_i \equiv \frac{\partial L}{\partial \dot{q}_i}}$ Also called generalized momentum

$$\boxed{\frac{d}{dt} p_i = \frac{\partial L}{\partial q_i}}$$

Big idea: For every **coordinate**, there is an accompanying **momentum**

Side comment on Quantum Mechanics

Big idea from Lagrangian Mechanics:
For every **coordinate**, there is an accompanying **momentum**

In QM,

$$q_i \rightarrow \hat{q}_i \rightarrow q_i$$

$$p_i \rightarrow \hat{p}_i \rightarrow \frac{\hbar}{i}\frac{\partial}{\partial q_i}$$

Substitute into the Hamiltonian

with p_i being the momentum accompanying q_i through $p_i = \frac{\partial L}{\partial \dot{q}_i}$

Simplest Example

$$L = K - V$$

K takes on the form $\frac{1}{2}m\dot{x}^2$

then $\frac{\partial L}{\partial \dot{x}} = p_x = m\dot{x} = mv$ (the familiar linear momentum)

Harmonic Oscillator:

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$$

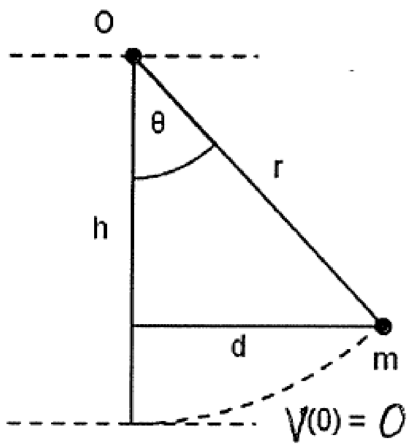
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \Rightarrow \frac{dp_x}{dt} + kx = 0 \Rightarrow m\frac{d^2x}{dt^2} + kx = 0$$

Systematic way of getting Equation of motion

“Simple” Pendulum: A not-so-simple example

[Physics is characterized by principles that can be generalized]

Coordinate is ϴ (doesn’t look like a coordinate in usual sense)



$V(\theta) = mg(r-h)$
$= mgr(1-\cos\theta)$

$V(0) = 0$

$$v = r\dot{\theta} \Rightarrow K = \frac{1}{2} m r^2 \dot{\theta}^2$$

$$V(\theta) = mgr(1-\cos\theta)$$

$$\boxed{L(\theta, \dot{\theta}) = \frac{1}{2} m r^2 \dot{\theta}^2 - mgr(1-\cos\theta)}$$

$$\frac{\partial L}{\partial \dot{\theta}} = \text{momentum conjugated to coordinate } \theta = m r^2 \dot{\theta} = r(m r \dot{\theta})$$

$= \text{angular momentum}$
(Comes out automatically)

$$\frac{\partial L}{\partial \theta} = -mgr\sin\theta \quad (\text{torque})$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \Rightarrow m r^2 \ddot{\theta} + mgr\sin\theta = 0 \Rightarrow \boxed{\ddot{\theta} = -\frac{g}{r}\sin\theta}$$

Key points: Angular momentum and torque appear by themselves, and systematic way of getting equation of motion!

Good Stuff #3:

- Easy identification of ***conserved quantities***

Simplest Example:

$$L = \frac{1}{2} m \dot{x}^2 \quad \text{(free particle)}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \Rightarrow \frac{dp_x}{dt} = 0 \quad \text{or } p_x \text{ is a constant in time}$$

$[p_x \text{ is conserved}]$

What's being done?

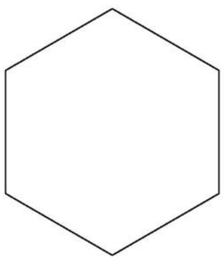
- If L does not depend on a coordinate x, then the (generalized) momentum p_x does not change in time (conserved)!

Deeper thought:

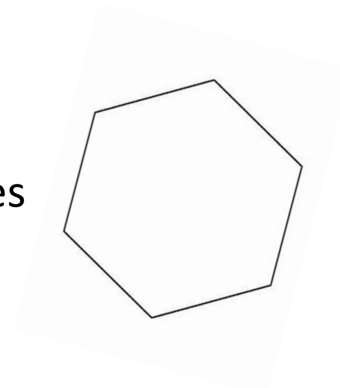
- **Symmetry and Conservation Law**

Symmetry? What symmetry?

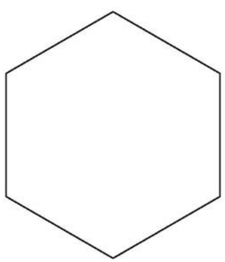
- Take something (an object such as a hexagon)
- Do something (operation) (rotation about the center through some angle)
- Has it been changed? If not, there is a symmetry of the object for the operation



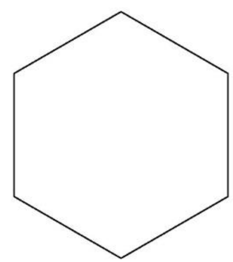
Rotated by 10 degrees



Changed!



Rotated by 60 degrees



Not changed! There is a symmetry!

But what is the "object" in Mechanics that we want to study its symmetry?

The "object" is the Lagrangian L

$$L = \frac{1}{2} m \dot{x}^2 \quad \text{(free particle)}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \Rightarrow \frac{dp_x}{dt} = 0 \quad \text{or } p_x \text{ is a constant in time}$$

$$[p_x \text{ is conserved}]$$

Think like a physicist!

- Free particle => no force => V = 0 (or same constant) everywhere
- Thus, space *x* is homogeneous (meaning the same everywhere)
- Fancy way of saying something simple:
    - Make an arbitrary ***translation in space*** *x -> x + δ (called a transformation)*
    - Inspect L and see if L is changed or not
    - **If L does not change** under the transformation (translation here), then obviously the same equation of motion follows (physics hasn't been changed)
    - Accompanying this symmetry (L does not change and same eq of motion), **something is conserved** (p_x here)

Bonus: Why bother and why so fancy?

- Big stuff in 20^{th} century physics!
- Jumping to about 1930
- E.g. Requiring *L* for an electron not to change under a transformation of the phase of the quantum mechanical wavefunction ("phase (gauge) transformation") of the electron, the correct formula describing how an electron interacts with EM fields appears automatically! (This is QED and it is a gauge field theory (Yang and Mills 1954)).

Good Stuff #4:

- ***Principle of Least Action (completed by Hamilton)***

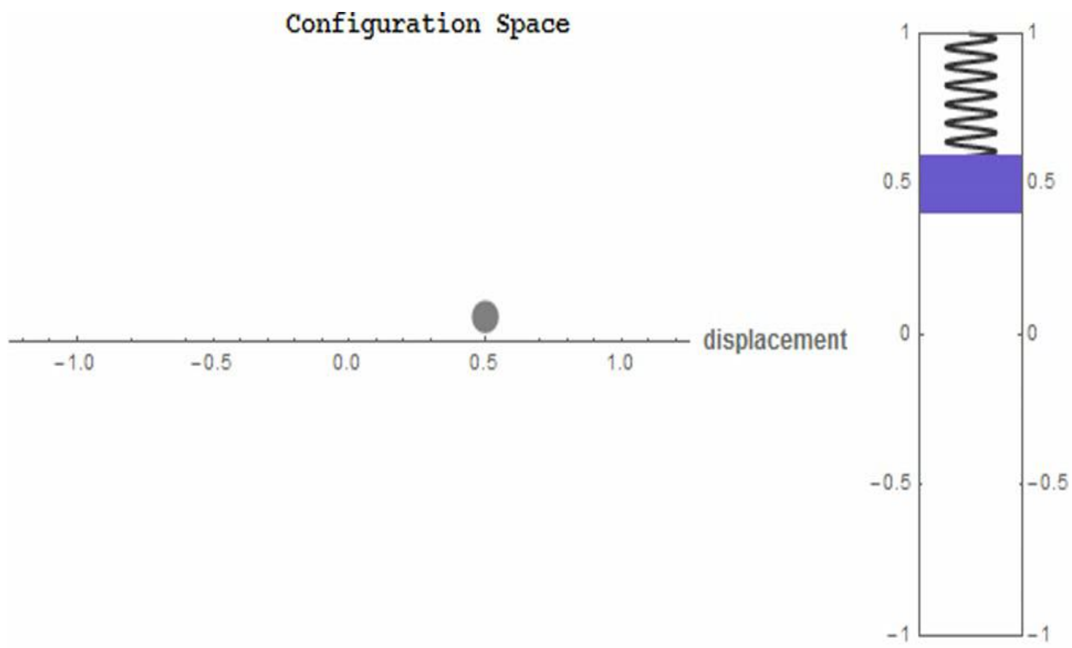
A way to interpret the Lagrange equation $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)-\frac{\partial L}{\partial x}=0$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)-\frac{\partial L}{\partial x}=0$ gives the correct $\underbrace{\text{equation of motion (Newton's law)}}_{\text{needs two conditions}}$

- Conditions: $\underbrace{(x_0, t_0)}_{\text{initial}}$ and $\underbrace{(x_1, t_1)}_{\text{final}}$ [instead of (x_0,t_0), (v_0,t_0)]
- Eq. of motion or $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)-\frac{\partial L}{\partial x}=0$ gives the actual trajectory of how the system goes from (x_0,t_0) to (x_1,t_1)
  [trajectory in configuration space defined by x]
- Anything special about the actual trajectory? (This is the question)

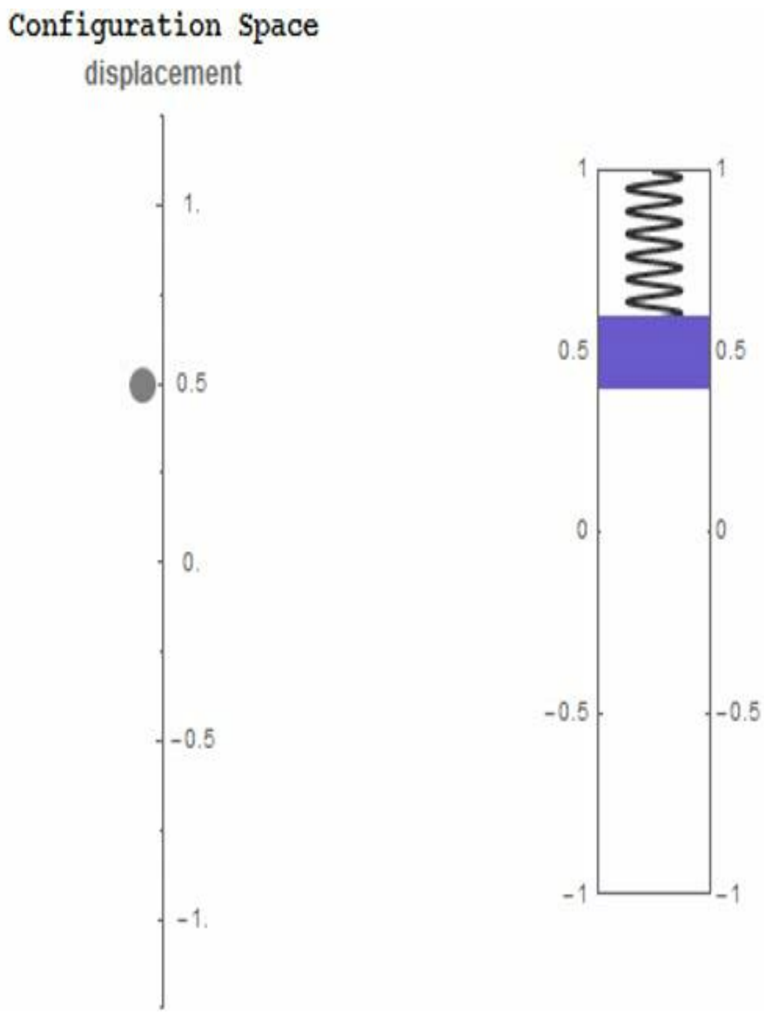
Harmonic oscillator (Configuration Space, displayed horizontally)

Lagrangian Mechanics is about the **configuration space**, which is just *x* (a line for 1D problem). The Euler-Lagrange equation (equation of motion) governs how the system evolves in configuration space in time, thus just *x(t)*.

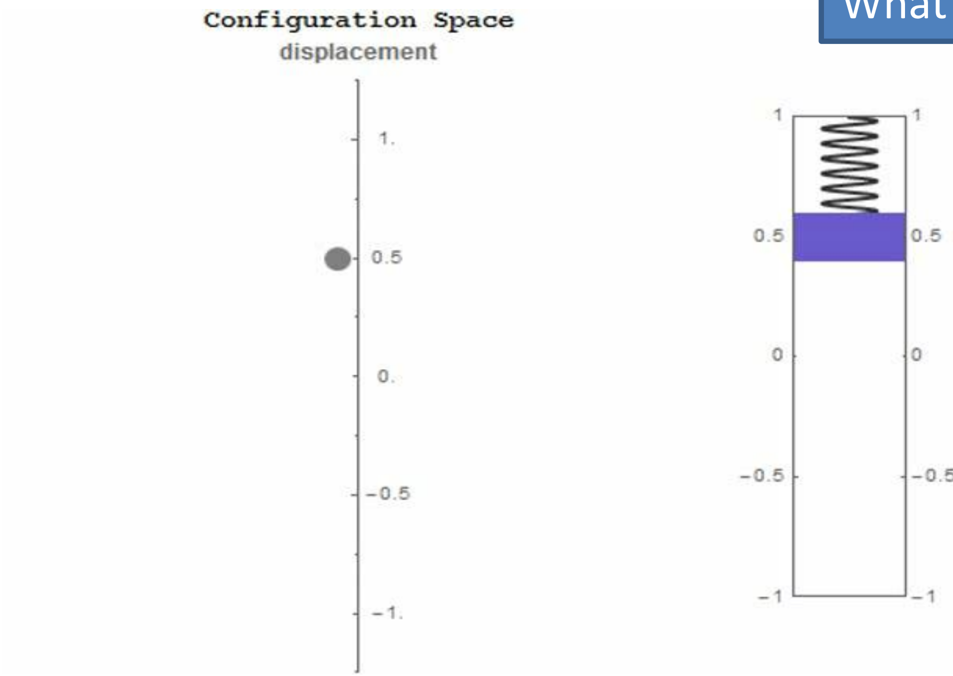


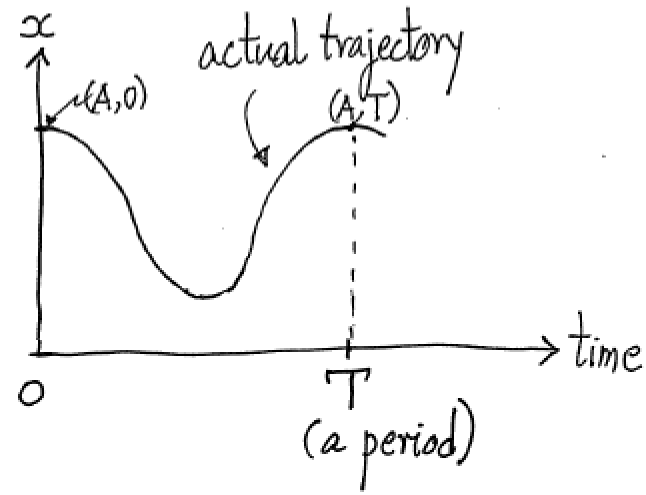
Harmonic oscillator (Configuration Space, displayed vertically)

Lagrangian mechanics is about the **configuration space**, which is just *x* (a line for 1D problem). The Euler-Lagrange equation (equation of motion) gives how the system evolves in configuration space in time, thus just *x(t)*, here displayed vertically.



What is so special about the actual trajectory?





[Maupertuis (1747), d'Alembert (1743), Euler and Lagrange (1750's), Hamilton (1834)]

They recognized that the trajectory following $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$ is the one that minimizes the quantity

$$S = \int_{t_0}^{t_1} L(x, \dot{x})\, dt$$ (The Least Action Principle)

Why taken so long (90 years)? Need to wait for the maths (calculus of variation)

An illustration: Harmonic oscillator (actions for correct & wrong motions)

[Meaning of Least Action Principle: If not following the right trajectory, Action will be higher]

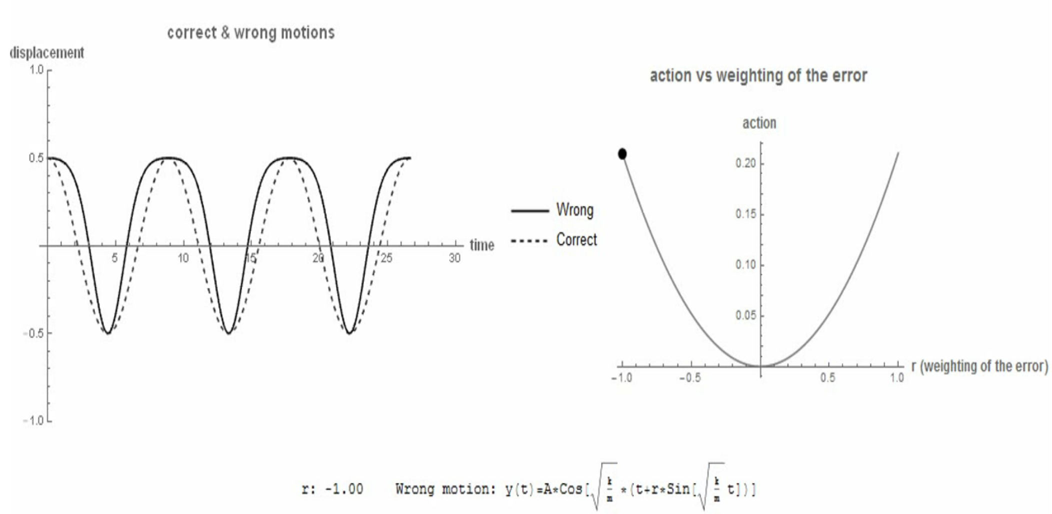
- Action for 3 periods of oscillation vs the weighting of the error function r

- Wrong Motion:

$$y(t) = A\cos\left(\sqrt{\frac{k}{m}}\left(t + r\sin\left(\sqrt{\frac{k}{m}}t\right)\right)\right)$$

- r (the weighting of the error function):
  from -1 to 1
- r = 0 -> correct motion

Dashed line = actual path



[Animation credit: LEUNG Chun Hei (MPhil Student, CUHK)]

Turning Newton's Law and Euler-Lagrange Equation into an ***extremum principle*** is profound!

- Principle of least time in ***Optics*** (Fermat 1657) – path of light from one point to another is one that takes the *shortest* time

- ***Mechanics*** follows suit (around 1750) – actual trajectory is one that *minimizes* the Action (inside action is the Lagrangian *L*)

- Thermodynamics and ***Statistical Physics*** – Entropy in a closed system is *maximized* as system approaches equilibrium. At constant pressure, reactions go in a way to *minimize* the Gibbs free energy *G*(T,p); etc.

- Formulating quantum field theories

Key ideas/Summary: Lagrangian Mechanics

- Getting equation of motion systematically from *L*
- For every coordinate, there is a momentum
- Easy to explore conservative laws and their relation to symmetry in *L*
- Put mechanics into an extremum principle (Least Action Principle) [Feynman formulated QM based on the action in 1950's]
- Bridging over to quantum mechanics: Coordinate-momentum pair(s) become operators in a systematic way
- Bridging over to Hamiltonian Mechanics

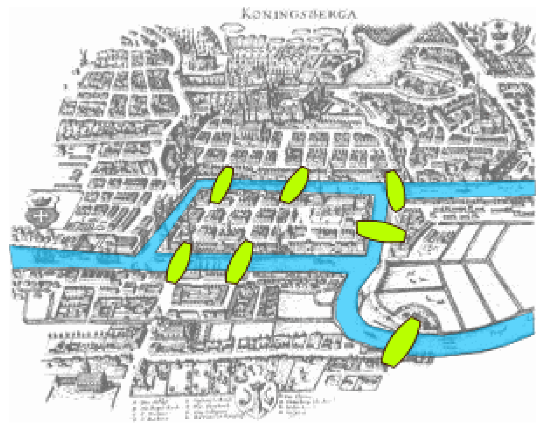
Let's meet Euler and Lagrange

Leonhard Euler (1707 – 1783)

Portrait by Jakob Emanuel Handmann (1753)
[Taken from Wikipedia]

$$\boxed{e^{i\pi} + 1 = 0}$$

"The most beautiful equation of mathematics"

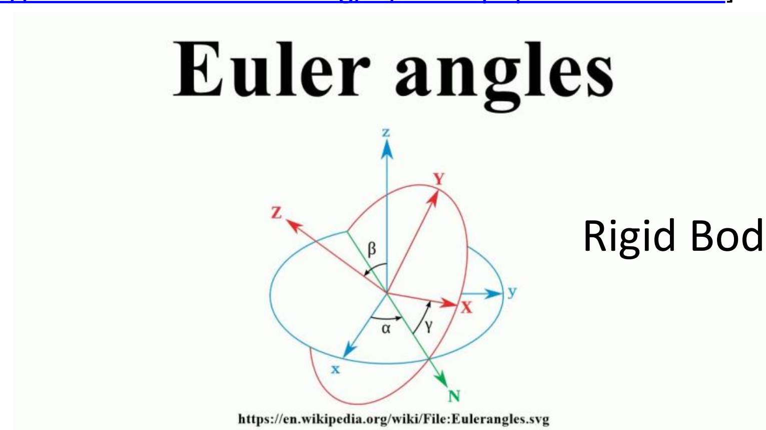


Graph Theory

[By Bogdan Giuşcă - Public domain (PD), based on the image, CC BY-SA 3.0, https://commons.wikimedia.org/w/index.php?curid=112920]

$$\rho \frac{\mathrm{Du}}{\mathrm{D}t} = \rho \left[\frac{\partial \mathrm{u}}{\partial t} + (\mathrm{u} \cdot \nabla)\, \mathrm{u} \right] = -\nabla p + \rho\, \mathrm{g}$$

Euler's equation in hydrodynamics



Rigid Body Motions

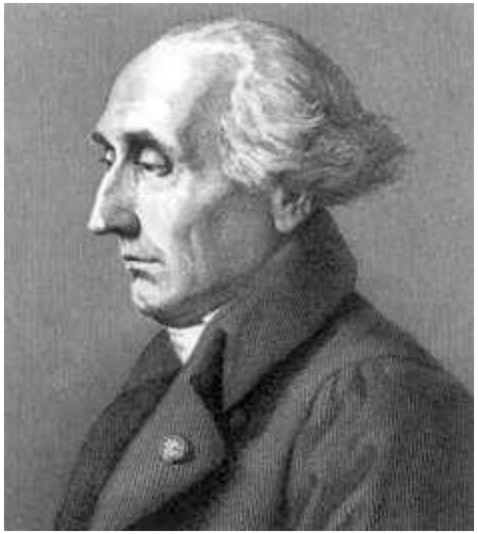
Leonhard Euler (1707 – 1783)

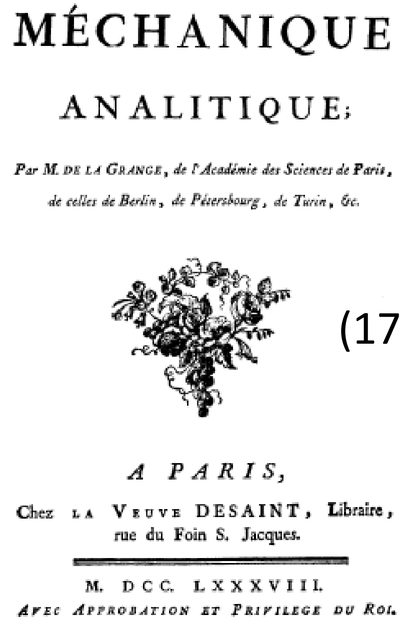
Euler is often regarded as the greatest mathematician of all time. He was born 1707 in Switzerland (29 years senior than Lagrange). At age 14, he entered the University of Basel for religious studies, but soon he found that he was talented in mathematics. He published many papers in mathematics when he was a student. At age 19 (1726), he completed university studies and immediately after that he was offered a position at the St. Petersburg Academy of Science in Russia. In St. Petersburg, he was surrounded by many gifted scientists and he worked and contributed to every branch of mathematics, pure and applied. In 1735 (28 years old), Euler lost the vision in one eye due to a serous fever, but he remained productive. In Russia, Peter the Great died in 1725 and Catherine the Great would not become Empress until 1762. Russia was politically unstable in the 1730's. Euler moved to the Berlin Academy of Science in 1741 at the invitation of Frederick the Great (King of Prussia) and worked there for 25 years. During those years, he published close to 400 articles. During his stay in Berlin, Euler invited Lagrange (29 years younger than him) to join him in Berlin in 1755 when Lagrange was only 19 years old. But Lagrange turned down the offer and preferred to work in his home town instead.

Leonhard Euler

In 1766, Russian had stabilized after Catherine the Great became Empress, and Euler went back to St. Petersburg. A few years later, he lost the vision of his another eye, but his mathematical works continued with the help of his two sons and assistants. Euler made many contributions. In physics, you see his Euler angles for rigid body, his equation in hydrodynamics, and in mechanics. He invented the calculus of variations. In the least action principle, his method helped when we varied the path from the actual trajectory. This is what nowadays called functional derivatives. Euler, D'Alembert and Lagrange defined the subject of Analytic Mechanics. Euler also worked on vibrations of strings and membranes and he interpreted light as waves. He died in 1783 in St. Petersburg. The Russian Academy of Sciences continued to publish his completed works for almost 40 more years after his death.

Joseph Louis Lagrange (1736 – 1813)





MÉCHANIQUE

ANALITIQUE;

Par M. de la Grange, de l'Académie des Sciences de Paris, de celles de Berlin, de Pétersbourg, de Turin, &c.

A PARIS,

Chez la Veuve DESAINT, Libraire, rue du Foin S. Jacques.

M. DCC. LXXXVIII.

Avec Approbation et Privilege du Roi.

(1787)

He participated in defining the SI units at the time of the French Revolution

Joseph-Louis Lagrange (1736 – 1813)

Lagrange was born 1736 in Turin (now in Italy). His family kept the French spelling of the surname although they had lived in Italy for generations. Lagrange wanted to be a lawyer, but his father lost his fortune and could not support him to do that. He studied at the University of Turin and discovered a talent in mathematics. At age 19, Lagrange become Professor of Mathematics at the Turin Royal Artillery School. At that time, he was so good that Euler invited him to Berlin to join him. But Lagrange preferred to work alone in his home town. His early contributions to physics and mathematics were about the theories on vibration of strings and propagation of sound. He eventually moved to the Berlin Academy of Science in 1766 to occupy Euler's position, when Euler moved back to St. Petersburg. Lagrange worked in Berlin for 20 years. When the political climate turned bad in Berlin, he moved to the Academie des Sciences (Academy of Science) in Paris in 1787 where he published his important books on Analytic Mechanics, which transformed Mechanics into a branch of mathematical analysis, and Analytic Functions. Later, he taught at Ecole Normale and Ecole Polytechnic in Paris.
You should have seen the Lagrange multipliers in problems for maximizing a function under some constraints. It is again this technique that has led to the Least Action Principle and the Euler-Lagrange Equation.

Joseph-Louis Lagrange

1790's was an exceptional period in France. It was the end of the Age of Enlightenment and the beginning of the French Revolution. A cause of the French Revolution, believe it or not, was the inconsistency in the measurement system. There were great scientists in France at that time – Laplace and Legrendre for example. Lagrange's personality avoided him to get into the political conflicts at that time, and yet he was involved in the effort of defining then new Metric System, which is still in use today. A note on the SI units. The definition of Kg, which Lagrange worked on in 1790s, may be changing later this year to one that replies on the Planck constant. Somehow quantum physics gets into the new definition of the kilogram! This is a great example about the Nature of Science. The Eiffel Tower in Paris has 72 scientists' names engraved on it. Lagrange is one of them. The next time you go to Paris, find Lagrange and Fourier there and take a picture.

Hamiltonian Mechanics

We saw the law of conservation of energy in Newtonian Mechanics

From how *L* not depending on *coordinates*, we have many *conservation laws* (often involving *momenta*).

Question: *Where is conservation of energy in Lagrangian Mechanics*?

Answer: The key points are whether *L* depends on time *t* explicitly and what energy really is

Hamilton (1805 – 1865)   Lagrange (1736 – 1813)

Getting something from nothing: Let's do 3 steps of math

Simple Case: $L(x, \dot{x}, t)$ [one coordinate]

- Look at $\frac{dL}{dt}$

$$\frac{dL}{dt} = \underbrace{\frac{\partial L}{\partial x}}\dot{x} + \underbrace{\frac{\partial L}{\partial \dot{x}}}_{p}\ddot{x} + \frac{\partial L}{\partial t} \quad \text{(calculus)}$$

Lagrange Equation $\rightarrow$ $\frac{dp}{dt} = \dot{p}$ (Euler-Lagrange Eq.)

$$= \underbrace{\dot{p}\dot{x} + p\ddot{x}} + \frac{\partial L}{\partial t}$$

$$\Rightarrow \frac{dL}{dt} = \frac{d}{dt}(p\dot{x}) + \frac{\partial L}{\partial t} \quad \text{(calculus)}$$

$$\Rightarrow \frac{d}{dt}\underbrace{(p\dot{x} - L)}_{\text{"something" (an energy, call it } H)} = -\frac{\partial L}{\partial t} \quad \text{(moving terms)}$$

$$\Rightarrow \boxed{\frac{d}{dt}H = -\frac{\partial L}{\partial t}} \quad \text{(General Statement)}$$

$$\boxed{H \equiv p\dot{x} - L} \quad \text{"Hamiltonian"}$$

Here enters the Hamiltonian.
The root of Hamiltonian in QM

Getting something from nothing: Big Ideas

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t}$$

- *H* has dimension energy
- If *L* does not depend on *time* explicitly, then *H* is conserved (does not change in time)!
- This is a statement of ***conservation of energy***
- Energy conservation is related to symmetry of *L* in translation in time
- But *what is energy*?

$$H = p\dot{x} - L$$

- Derived (identified) *H*
- *H* is constructed from quantities in Lagrangian Mechanics
- Harmonic oscillator

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$$

$$p = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

$$\begin{aligned} H &= p\dot{x} - L \\ &= m\dot{x}^2 - \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 \\ &= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 \\ &= \text{total energy } E \\ &= \frac{p^2}{2m} + \frac{1}{2}kx^2 \end{aligned}$$

Total energy

Important Features of Hamiltonian *H*

- To get *H*, we need *L* (thus identified coordinates), then find generalized momenta, then construct *H* (thus need to learn Lagrangian Mechanics first)

- *H(x,p)* is a function of coordinate-momentum pair(s)

momentum conjugated to coordinate

$$H(x,p) = \frac{p^2}{2m} + \frac{1}{2}kx^2$$

coordinate-momentum pair

coordinate

For students prefer practicality, simply take H(x,p) = total energy
= K.E. + P.E
and start doing QM

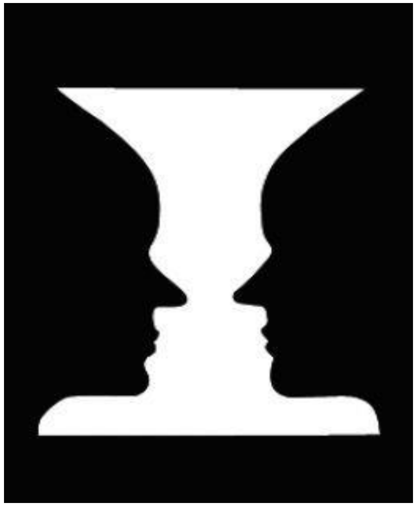
Why so? (next page)
Why is it important? The Hamiltonian (plus operators) is the starting point of Quantum Mechanics

Why so?

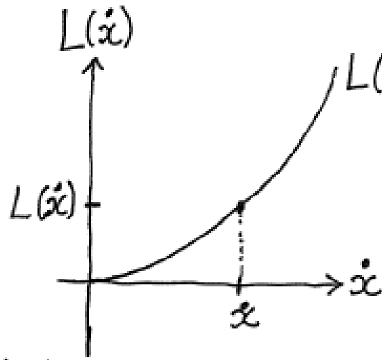
$L(x, \dot{x})$ or $H(x, p)$?

$L(\dot{x})$ or $H(p)$?

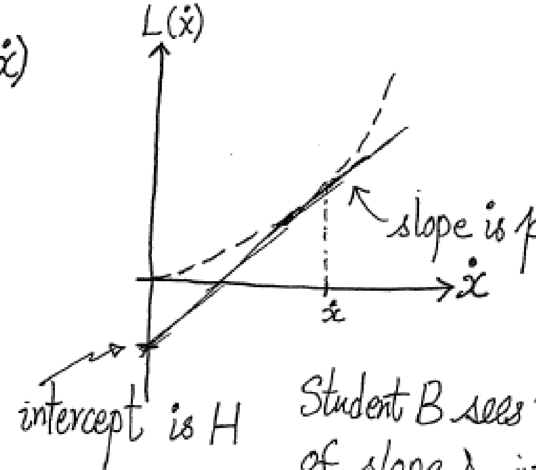
Vase or Faces?



Two ways of looking at the *Same Picture*



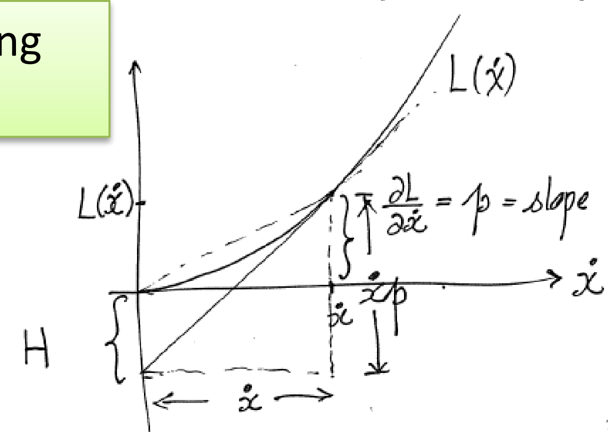
Student A sees $L(\dot{x})$, i.e. values of $\dot{x}$ & $L(\dot{x})$



Student B sees values of slope & intercepts

The point is: Student A and Student B are conveying the same information, only in different ways!

This change in viewpoint is called a Legrendre (1752 – 1833) Transform. The same idea takes us from the internal energy U(S,V) to Helmholtz free energy F(T,V) or Gibbs free energy G(T,p) in thermodynamics.



Why is it important?

Jumping to 1925-1926 (Quantum Mechanics)

Harmonic Oscillator

momentum conjugated to coordinate

$$H(x,p) = \frac{p^2}{2m} + \frac{1}{2}kx^2$$

coordinate-momentum pair

coordinate

Go Quantum!

$$\hat{H}\psi = E\psi$$

Schrodinger Equation (1926) was built on the Hamiltonian

coordinate → coordinate operator $\hat{x}$ $\hat{x} \rightarrow x$

momentum → momentum operator $\hat{p}$ $\hat{p} \rightarrow -i\hbar\frac{\partial}{\partial x}$

Do this for coordinate-momentum pair

coordinate [whatever it is]

$$\left[\frac{-\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}kx^2\right]\psi(x) = E\psi(x)$$

Schrodinger Equation of a ***Quantum*** Harmonic Oscillator that gives the allowed energies *E* and wavefunctions $\psi(x)$

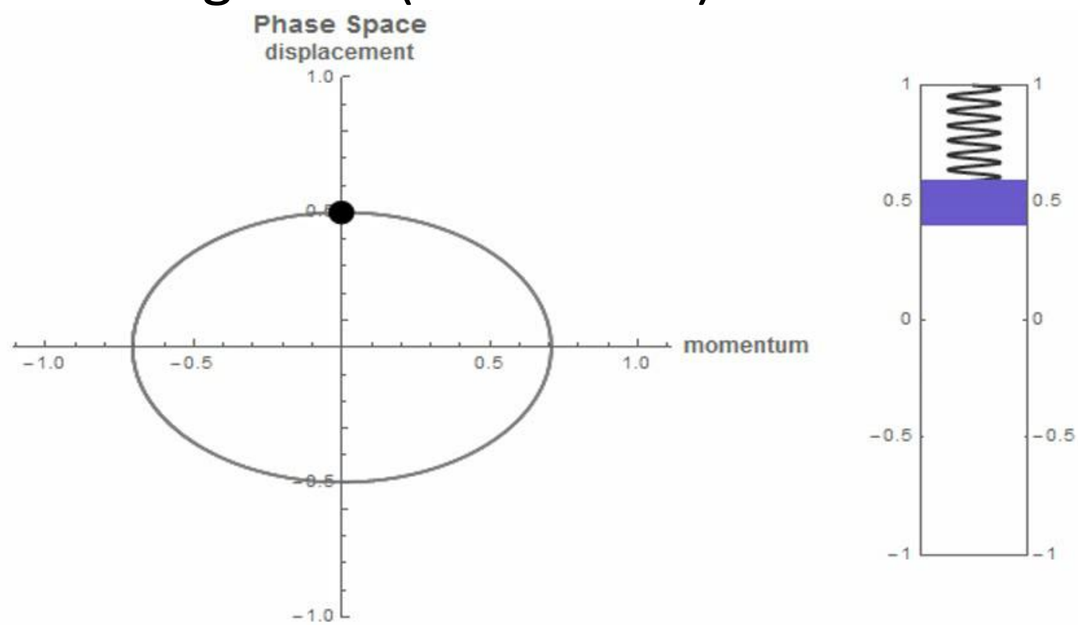
Why is it important?

Go back to Hamilton's time

H(x,p) describes a system by its ***coordinate and momentum*** as it evolves in time

Hamiltonian Mechanics introduces the ***Phase Space*** (x,p) (2D phase space) for a particle moving in 1D (x-direction)

Harmonic Oscillator
Amplitude A = 0.5
Spring constant k = 1
Mass m = 2
Energy = $\frac{1}{2}kA^2$
Conservation of energy =>
System confined to an ellipse in phase space



[Credit: LEUNG Chun Hei (MPhil student, CUHK)]

Equations governing motion in Phase Space - The Hamilton's equations

Getting something from nothing again!

$H(x,p)$ Thus $\delta H = \frac{\partial H}{\partial x}\delta x + \frac{\partial H}{\partial p}\delta p$ in general $(*)$

But $H = \dot{x}p - L$; $\delta H = p\,\delta\dot{x} + \dot{x}\,\delta p - \delta L$

$$= \cancel{p\,\delta\dot{x}} + \dot{x}\,\delta p \overbrace{- \underbrace{\frac{\partial L}{\partial x}}_{\dot{p}}\delta x - \cancel{\underbrace{\frac{\partial L}{\partial \dot{x}}}_{p}\delta\dot{x}}}$$

Lagrange Equation

$$= \dot{x}\,\delta p - \dot{p}\,\delta x \quad (\text{c.f. } *)$$

$$\therefore \boxed{\dot{x} = \frac{\partial H}{\partial p} \; ; \; \dot{p} = -\frac{\partial H}{\partial x}}$$

Hamilton's Equations of motion giving evolution in phase space

Two 1st order differential equations (c.f. Newton's law which is one 2nd order equation) giving how *x* updates and how *p* updates – they combine to give the equation of motion. Just another way of doing Mechanics!

Why is it important?

- Concept of phase space is fundamental in developing Statistical Mechanics (Boltzmann and Gibbs, end of 19^{th} century)
- Take the phase space with you to thermal physics & statistical mechanics courses

Further Development: More Classical Mechanics

$F(x, p)$ = some physical quantity

- How does F change as system (i.e. (x, p)) evolves?

Hamilton

x moves in phase space
p moves in phase space → F changes

$$\frac{dF}{dt} = \frac{\partial F}{\partial x} \cdot \dot{x} + \frac{\partial F}{\partial p} \cdot \dot{p} \quad \text{[partial derivatives]}$$

$$= \frac{\partial F}{\partial x}\frac{\partial H}{\partial p} - \frac{\partial F}{\partial p}\frac{\partial H}{\partial x} \quad \text{[Hamilton's equations]}$$

$$\Rightarrow \frac{dF}{dt} \equiv \{F, H\} \quad \text{"Poisson Bracket"}$$

Poisson (1781 – 1840)

In Heisenberg's quantum mechanics (1925), quantum operators evolve in time following an equation almost of the same form, with Bracket replaced by **commutator.**

Further Development

Poisson Bracket

Some functions $F(x,p)$, $G(x,p)$ [phase space involved]

$$\{F,G\}_{PB} \equiv \frac{\partial F}{\partial x}\cdot\frac{\partial G}{\partial p} - \frac{\partial F}{\partial p}\cdot\frac{\partial G}{\partial x}$$

Most important one is $\underbrace{\{x,p\}_{PB} = 1}_{\text{as close to quantum mechanics as classical mechanics can be}}$ (From F = x, G = p)

Dirac did his version of Quantum Mechanics (1925) starting from here!

Key ideas/Summary: Hamiltonian Mechanics

- Energy conservation is related to symmetry of L in time translation
- Identified $H(x,p)$
- Phase space and how x and p moves (Hamilton's equations)
- Led to different formulations of quantum mechanics in 1925-26
- Led to developments in Statistical Mechanics in late 1800's

Let’s meet Hamilton

William Hamilton (1805 – 1865)

$$\delta \int_{t_1}^{t_2} L(q,\dot{q})dt = 0\,,$$

Hamilton’s principle of least action

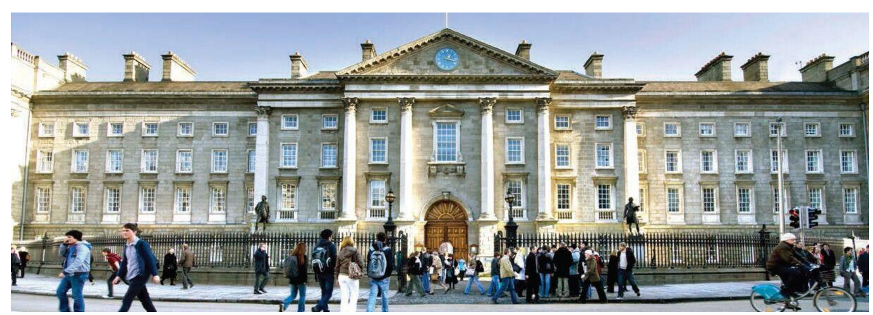
$$L(q,\dot{q}) = T(q,\dot{q}) - V(q),$$
$$T(q,\dot{q}) = \text{kinetic energy}\,.$$

$$\dot{q}^j = \frac{\partial H(q,p)}{\partial p_j}\,, \quad \dot{p}_j = -\frac{\partial H(q,p)}{\partial q^j}$$

Hamilton’s canonical equations of motion

$$\frac{\partial S(q,t)}{\partial t} + H\left(q, \frac{\partial S(q,t)}{\partial q}\right) = 0$$

Hamilton’s theory of optics has an equation highly similar to the Schrodinger Equation



Hamilton Mathematics Institute, Trinity College Dublin

William Hamilton (1805 – 1865)

Hamilton was born in 1905, and lived in Dublin, Ireland, all his life. He would be called a highly gifted person nowadays. His early talent was in linguistics. He was fluent in English at age 3. At age 5, he translated texts from Latin, Greek, and Hebrew. At age 13, he could read 13 languages. He then turned his focus to mathematics. At age 16, he worked through Newton's *Principia* and Laplace's *Mecanique Celeste* all by himself. At 18, he entered Trinity College in Dublin and won outstanding awards in both the classics and the sciences. He worked on mathematical optics in his undergraduate years. He presented his results in *Theory on Systems of Rays* to the Royal Irish Academy in 1824 when he was only 19 years old, although the paper in published form was delayed to for 4 years and appeared in 1828. This early work was later regarded as a masterpiece as it showed a way that optics and mechanics could be formulated in the same way mathematically. This Hamilton's optics has an equation that looks very much like the Schrodinger equation in quantum mechanics developed about a century later. In fact, Schrodinger referred to Hamilton's work in his QM papers.

William Hamilton

Hamilton was appointed Professor of Astronomy at Dublin before his graduation at age 22, a position that he held to his death at age 60. He was also appointed the Astronomer Royal of Ireland, and even better he was allowed not to carry out the duties and left alone to work on his theories! In 1834 (age 29), he published the classic paper "*On a general method on dynamics*", which is 62 pages long. He formulated classical mechanics in a way that would serve as one of the principal formulations of quantum mechanics in the works of Heisenberg and Dirac (some 90 years later). Therefore, his work influenced all the founders (Schrodinger, Heisenberg, and Dirac) of quantum mechanics. There are many things under his name -- the Hamiltonian, which we use a lot in quantum mechanics, his final version of the principle of least action, and the Hamiltonian canonical equations. Later in his life, he worked on turning complex numbers into a subject with solid algebraic foundation. He started a related subject call Quaternions and wrote a 800-page book on it, but the subject has not found its place in mathematics so far. He also consumed too much alcohol in his later years.

Summary: Classical Mechanics goes far beyond its domain!

Minimal learning outcomes for this discussion

An appreciation that classical mechanics had led a good foundation for physicists around 1900 to pursue quantum mechanics

Key ideas (think/talk like a physics student): Newton's eq. of motion -> Lagrangian + Euler equation also give eq. of motion -> Lagrangian enters -> define momentum for each coordinate (coordinate – momentum pair) [QM needs them] -> Hamiltonian [QM starts with H(x,p)] -> Hamilton's mechanics deals with phase space -> motion in phase space implies Poisson bracket [closely related to QM commutator]

[These are ideas to know, not the technical details!]

Practically:
All you need to know is to write down the Hamiltonian

$$H(x,p) = KE + PE = T + U$$

for a given system,
and to realize that there is a classical mechanics root of the Hamiltonian operator in QM

Big Picture – Paths to perfection of Mechanics after Newton

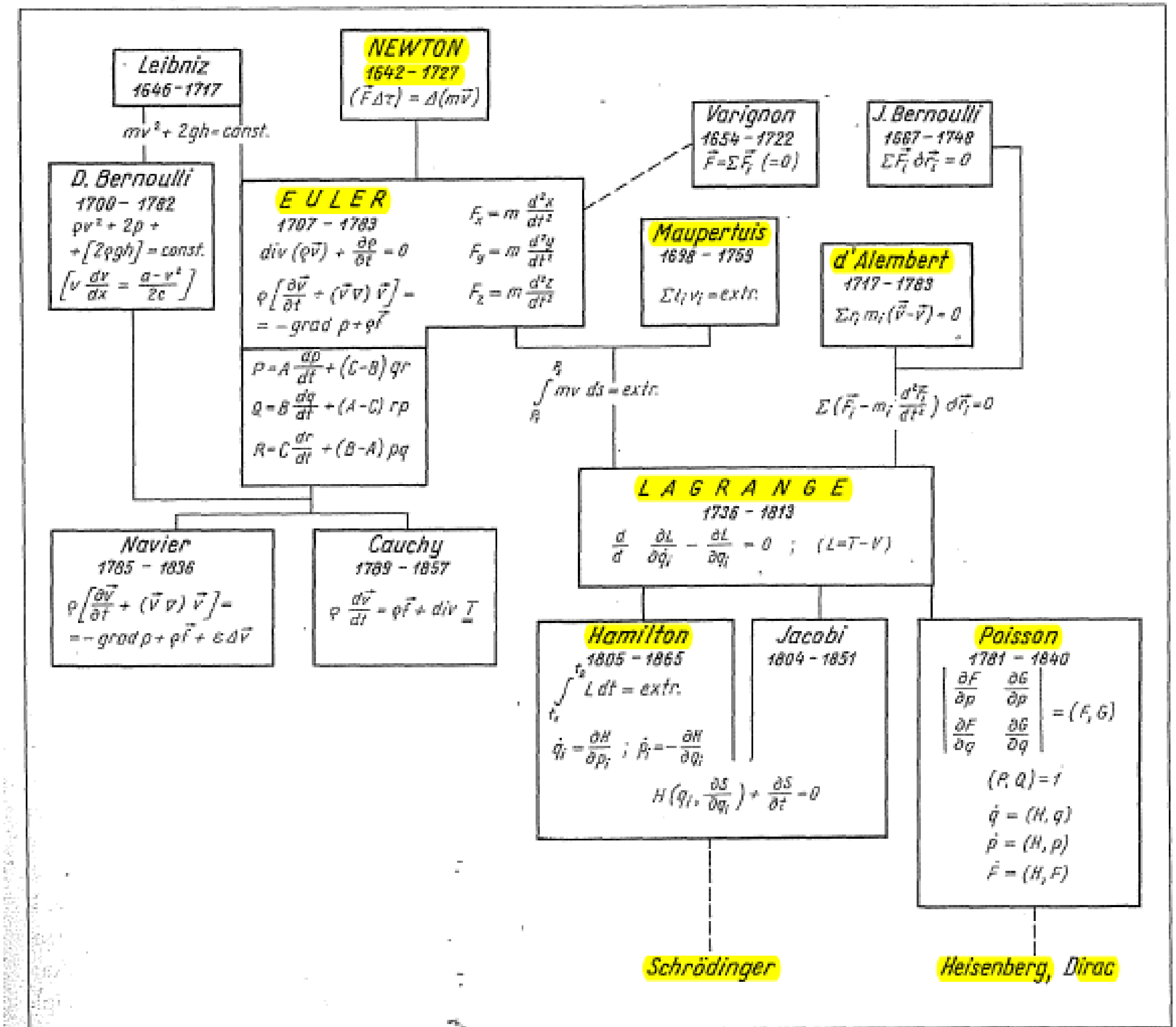


Figure 4.31 Paths to the perfection of mechanics after Newton.

We shall have more to say later in the text on the equations of motion for systems of mass points and of rigid bodies.

The mechanics of continua begins with the Bernoulli and Euler equations. These relate, respectively, to incompressible and frictionless ideal fluids. Navier proposed his equation in 1822, in which he considers internal friction using the coefficient of viscosity ε. This equation is usually referred to as the Navier–Stokes equation, although Stokes presented the equation in a more general form in 1845. Cauchy's equation describes the motion of deformable solid bodies. Here T is the stress tensor. Turbulent flow was investigated by Osborne Reynolds (1842–1912; introduction of the Reynolds number, which measures the presence of turbulence), Ludwig Prandtl (1875–1953; theory of interfaces), and Theodore von Kármán (1881–1963). Today, these topics are of great interest: Chaos, a new scientific discipline, provides a method for treating problems in turbulent flow.

Taken from: K. Simonyi, *A Cultural History of Physics* (CRC Press 2010)

I hope that you see what you learned in previous courses is so profoundly *uselessly* (you might have thought) *useful*!